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A Thick Plate Problem Under the Action of a Body Force in Generalized Thermoelasticity

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Abstract The two-dimensional problem for a thick plate is considered within the context of the theory of generalized thermoelasticity with one relaxation time under the action of a body force. The upper surface of the plate is subjected to a known temperature distribution, while its lower one is laid on a thermally insulated rigid foundation. Laplace and exponential Fourier transform techniques are used. The solution in the transformed domain is obtained by a direct approach. The inverse double transform is evaluated numerically. The distributions of the considered physical variables are obtained and represented graphically.

Keywords Body forces · Exponential Fourier transform · Generalized thermoelasticity · Laplace transform · Thermal relaxation times · Thick plate

1 Introduction

The classical theory of thermoelasticity predicts an infinite speed for heat propagation, Biot [1], which is contrary to physical observations. To overcome this paradox, many papers have been devoted to the development of the generalized theory of thermoelasticity that predicts a finite speed for heat propagation.

The generalized theory of thermoelasticity developed by Lord and Shulman (L–S) [2] is based on a modified Fourier's law whose governing system of equations is entirely hyperbolic and hence predicts a finite speed for heat propagation. Dhaliwal and Sherief [3] extended this theory to include the anisotropic case. In this theory, a modified

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law of heat conduction including both the heat flux and its time derivative replaces the conventional Fourier's law. The uniqueness of a solution for this theory was proved under different conditions by Ignaczak [4,5] and by Sherief [6]. Sherief and El-Maghraby [7,8] have solved two crack problems. Mallik and Kanoria [9] have formulated a generalized thermoelasticity application for a penny-shaped crack analysis. Abdel-Halim and Elfalaky [10] solved a problem for an internal penny-shaped crack in an infinite thermoelastic solid. Elfalaky and Abdel-Halim [11] solved a two-dimensional problem for an infinite space weakened by a finite linear opening Mode-I crack. Mallik and Kanoria [12] have solved a two-dimensional problem for a transversely isotropic generalized thick plate with spatially varying heat. A two-dimensional problem for a half-space with a heat source has been solved by El-Maghraby [13]. A one-dimensional problem for a half-space under the action of a body force has been solved by Saleh [14].

In this work, we investigate a two-dimensional problem for a thick plate made of a homogeneous isotropic thermoelastic solid. The upper surface of the plate is subjected to a known temperature distribution, while the lower surface is laid on a thermally insulated rigid foundation. The problem is in the context of the generalized theory of thermoelasticity with one relaxation time and under the action of a body force. Laplace and exponential Fourier transform techniques are used. The solution in the transformed domain is obtained by a direct approach. The inverse double transform is evaluated numerically. Numerical results for the distributions of temperature, horizontal displacement, and normal stress for various times are obtained, and represented graphically.

2 Formulation of the Problem

We consider a homogeneous isotropic thermoelastic solid occupying the region $-l \leq x \leq l$. The x-axis is taken perpendicular to the bounding planes. We also assume that the initial state of the medium is quiescent. The upper surface of this medium is subjected to a known temperature distribution $f(y, t)$. The lower surface is laid on a rigid foundation and is thermally insulated. The displacement vector thus has the form:

$$\underline{u} = (u, v, 0)$$

and the cubical dilation e is given by

$$e = \nabla \cdot \underline{u} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \tag{1}$$

$$\sigma_{ij} = 2\mu e_{ij} + \left[\lambda e - \gamma \left(T - T_0\right)\right] \delta_{ij}$$

The governing equations of thermoelasticity with one relaxation time consist of [15]

- The equation of motion in thermo-vector form

$$\rho \frac{\partial^2 u}{\partial t^2} = (\lambda + \mu)\, \nabla e + \mu \nabla^2 u - \gamma \nabla T + \rho \underline{F} \tag{2}$$

which has two Cartesian components:

$$\rho \frac{\partial^2 u}{\partial t^2} = (\lambda + \mu)\, \nabla e + \mu \nabla^2 u - \gamma \nabla T + \rho F_x, \tag{3}$$

$$\rho \frac{\partial^2 v}{\partial t^2} = (\lambda + \mu)\, \nabla e + \mu \nabla^2 v - \gamma \nabla T + \rho F_y \tag{4}$$

where ρ is the density, t is the time variable, λ and μ are Lamé's constants, T is the absolute temperature, γ is a material constant given by $\gamma = (3\lambda + 2\mu)\, \alpha_t$, and α_t is the coefficient of linear thermal expansion.

- The generalized equation of heat conduction in the absence of heat sources [8]

$$k \nabla^2 T = \left(\frac{\partial}{\partial t} + \tau_0 \frac{\partial^2}{\partial t^2} \right) + (\rho c_{\mathrm{E}} T + \gamma T_0 e), \tag{5}$$

where k is the thermal conductivity of the medium, T_0 is a reference temperature assumed to be such that $|(T - T_0) / T_0| \ll 1$, c_{E} is the specific heat at constant strain, τ_0 is a constant with the dimension of time that acts as a relaxation time, F_x and F_y are the applied body forces in the x and y directions, respectively, and ∇^2 is the Laplace's operator.

Applying the div operator $(\nabla \cdot)$ to both sides of Eq. 2, we obtain

$$\rho \frac{\partial^2 e}{\partial t^2} = (\lambda + 2\mu)\, \nabla^2 e - \gamma \nabla^2 T + \rho \nabla \cdot \underline{F}. \tag{6}$$

The following constitutive relations supplement the above equations:

$$\sigma_{xx} = (\lambda + 2\mu)\, e - 2\mu \frac{\partial v}{\partial y} - \gamma\, (T - T_0), \tag{7a}$$

$$\sigma_{yy} = (\lambda + 2\mu)\, e - 2\mu \frac{\partial u}{\partial x} - \gamma\, (T - T_0), \tag{7b}$$

$$\sigma_{xy} = \mu \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right), \tag{7c}$$

$$q + \tau_0 \dot{q} = -k \nabla T, \tag{7d}$$

where σ_{ij} are the components of the stress tensor and $\underline{q}$ is the heat flux vector.

For convenience, the following dimensionless quantities are introduced:

$$x' = c_1 \eta x, \qquad y' = c_1 \eta y, \qquad t' = c_1^2 \eta t,$$
$$u' = c_1 \eta u, \qquad v' = c_1 \eta v, \qquad \tau_0' = c_1^2 \eta \tau_0,$$
$$\sigma_{ij}' = \frac{\sigma_{ij}}{\lambda + 2\mu}, \quad F_x' = \frac{\rho F_x}{(\lambda + 2\mu)\, c_1 \eta}, \quad F_y' = \frac{\rho F_y}{(\lambda + 2\mu)\, c_1 \eta}$$

where $\eta = \rho c_E/k$ and c_1 is the speed of propagation of isothermal elastic waves given by $c_1 = \sqrt{(\lambda + 2\mu)/\rho}$.

Using the above dimensionless variables and by dropping primes for convenience, the governing equations become

$$\beta^2 \frac{\partial^2 u}{\partial t^2} = \left(\beta^2 - 1\right) \frac{\partial e}{\partial x} + \nabla^2 u - \beta^2 \frac{\partial \theta}{\partial x} + \beta^2 F_x, \tag{8}$$

$$\beta^2 \frac{\partial^2 v}{\partial t^2} = \left(\beta^2 - 1\right) \frac{\partial e}{\partial y} + \nabla^2 v - \beta^2 \frac{\partial \theta}{\partial y} + \beta^2 F_y, \tag{9}$$

$$\nabla^2 \theta = \left(\frac{\partial}{\partial t} + \tau_0 \frac{\partial^2}{\partial t^2}\right) (\theta + \varepsilon e) \tag{10}$$

$$\frac{\partial^2 e}{\partial t^2} = \nabla^2 e - \nabla^2 \theta + \nabla \cdot \underline{F} \tag{11}$$

while the constitutive relations Eqs. 7a–7d become, respectively,

$$\sigma_{xx} = 2\frac{\partial u}{\partial x} + \left(\beta^2 - 2\right) e - \beta^2 \theta \tag{12a}$$

$$\sigma_{yy} = 2\frac{\partial v}{\partial y} + \left(\beta^2 - 2\right) e - \beta^2 \theta \tag{12b}$$

$$\sigma_{xy} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \tag{12c}$$

$$q_x + \tau_0 \frac{\partial q}{\partial t} = -\frac{\partial \theta}{\partial x} \tag{12d}$$

In the above equations, we have used the abbreviations,

$$\beta^2 = \frac{\lambda + 2\mu}{\mu} \quad \text{and} \quad \varepsilon = \frac{T_0 \gamma^2}{\rho c_E (\lambda + 2\mu)}$$

The above equations are solved and subjected to the initial conditions,

$$\theta = u = v = 0 \quad \text{and} \quad \frac{\partial u}{\partial t} = \frac{\partial v}{\partial t} = \frac{\partial \theta}{\partial t} = 0 \quad \text{at} \quad t = 0. \tag{13}$$

The boundary conditions for the problem are

$$\theta(l, y, t) = \theta_0 \, \mathrm{H}(c - |y|) \, \mathrm{H}(t), \tag{14a}$$

$$\sigma_{xx}(l, y, t) = 0, \tag{14b}$$

$$\sigma_{xy}(l, y, t) = 0, \tag{14c}$$

$$u(-l, y, t) = 0, \tag{14d}$$

$$v(-l, y, t) = 0, \tag{14e}$$

$$q_x(-l, y, t) = 0. \tag{14f}$$

In Eq. 14a, θ_0 and c are constants while H(·) is the Heaviside unit step function. Thus, the surface $x = l$ is heated on a band of width $2c$ on the upper surface, while the rest of the surface is kept at zero temperature.

We note here that Eq. 14a implies that [17]

$$\frac{\partial}{\partial t}\theta\,(l, y, t) = \theta_0\,\mathrm{H}\,(c - |y|)\,\delta\,(t),$$

where $\delta(\cdot)$ denotes the Dirac delta function. The above equation means that $\partial\theta/\partial t$ takes an infinite value at $x = l$, $t = 0$. This is a customary thermal shock problem.

3 Solutions in the Laplace Transform Domain

By applying the Laplace transform defined by the relation:

$$\overline{f}(x, y, s) = \mathrm{L}\,[f(x, y, t)] = \int_0^\infty \mathrm{e}^{-st} f(x, y, t)\mathrm{d}t,$$

to both sides of Eqs. 8–12c, we obtain, respectively,

$$\left(\nabla^2 - \beta^2 s^2\right)\overline{u} = \left(1 - \beta^2\right)\frac{\partial\overline{e}}{\partial x} + \beta^2\frac{\partial\overline{\theta}}{\partial x} - \beta^2\overline{F}_x, \tag{15}$$

$$\left(\nabla^2 - \beta^2 s^2\right)\overline{v} = \left(1 - \beta^2\right)\frac{\partial\overline{e}}{\partial y} + \beta^2\frac{\partial\overline{\theta}}{\partial y} - \beta^2\overline{F}_y, \tag{16}$$

$$\left(\nabla^2 - s - \tau_0 s^2\right)\overline{\theta} = \varepsilon\left(s + \tau_0 s^2\right)\overline{e}, \tag{17}$$

$$\left(\nabla^2 - s^2\right)\overline{e} = \nabla^2\overline{\theta} - \nabla\cdot\underline{\overline{F}}, \tag{18}$$

$$\overline{\sigma}_{xx} = \beta^2\overline{e} - 2\frac{\partial\overline{v}}{\partial y} - \beta^2\overline{\theta} \tag{19a}$$

$$\overline{\sigma}_{yy} = \beta^2\overline{e} - 2\frac{\partial\overline{u}}{\partial x} - \beta^2\overline{\theta}, \tag{19b}$$

$$\overline{\sigma}_{xy} = \frac{\partial\overline{u}}{\partial y} + \frac{\partial\overline{v}}{\partial x}. \tag{19c}$$

The Laplace transform of the cubical dilatation, Eq. 1, has the form,

$$\overline{e} = \frac{\partial\overline{u}}{\partial x} + \frac{\partial\overline{v}}{\partial y} \tag{20}$$

Also, the Laplace transform for Fourier's law of heat conduction, Eq. 12d, can be written as

$$\overline{q}_x = -\frac{1}{1+\tau_0 s}\frac{\partial \overline{\theta}}{\partial x}, \tag{21}$$

Eliminating $\overline{e}$ between Eqs. 17 and 18, we get

$$\left[\nabla^4 - \left(s^2(1+\tau_0) + s(1+\varepsilon)\right)\nabla^2 + s^3(1+\tau_0 s)\right]\overline{\theta} = -\varepsilon s \nabla \cdot \underline{\overline{F}} \tag{22}$$

In order to solve the above equation, we shall use the exponential Fourier transform with respect to y, which is defined by the relation [16,17],

$$\overline{f}^*(x,q,s) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \overline{f}(x,y,s)\,\mathrm{e}^{-\mathrm{i}qy}\mathrm{d}y.$$

The inverse transform is given by the relation [16,17],

$$\overline{f}(x,y,s) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \overline{f}^*(x,q,s)\,\mathrm{e}^{\mathrm{i}qy}\mathrm{d}q.$$

From now on, we shall take the heat source in the form,

$$F_x(x,y,t) = \frac{\mathrm{H}(t)\cosh bx}{y^2+a^2}, \qquad F_y(x,y,t) = 0,$$

where a and b are constants.

Thus, $\overline{F}_x(x,y,s) = \dfrac{\cosh bx}{s\left(y^2+a^2\right)}$, $\overline{F}_y(x,y,s) = 0$, and $\overline{F}^*_x(x,q,s) = \sqrt{\frac{\pi}{2}}\frac{\mathrm{e}^{-a|q|}}{sa}\cosh bx$, $\overline{F}^*_y(x,q,s) = 0$.

Applying the Fourier exponential transform to both sides of Eq. 22, we obtain

$$\left[\left(\mathrm{D}^2-q^2\right)^2 - \left(s^2(1+\tau_0)+s(1+\varepsilon)\right)\left(\mathrm{D}^2-q^2\right) + s^3(1+\tau_0 s)\right]\overline{\theta}^* = -\varepsilon s \mathrm{D}\overline{F}^*_x \tag{23}$$

where $\mathrm{D} = \dfrac{\partial}{\partial x}$.

The complementary function of the above equation is given by

$$\overline{\theta}^*_c = \left(k_1^2 - s^2\right)\left(A_1\mathrm{e}^{\mu_1 x} + A_2\mathrm{e}^{-\mu_1 x}\right) + \left(k_2^2 - s^2\right)\left(A_3\mathrm{e}^{\mu_2 x} + A_4\mathrm{e}^{-\mu_2 x}\right), \tag{24}$$

where $\mu_i^2 = k_i^2 + q^2, i = 1, 2, A_i = A_i(q,s), i = 1, 2, 3, 4$, are parameters depending on q and s and k_1^2 and k_2^2 are roots of the characteristic equation.

$$k^4 - \left[s^2(1+\tau_0) + s(1+\varepsilon)\right]k^2 + s^3(1+\tau_0 s) = 0$$

The particular solution of Eq. 23 can be written in the form,

$$\overline{\theta}_p^* = -\sqrt{\frac{\pi}{2}} \frac{\varepsilon b \mathrm{e}^{-a|q|} \sinh bx}{a\left(b^2-\mu_1^2\right)\left(b^2-\mu_2^2\right)}. \tag{25}$$

Using Eqs. 24 and 25, the general solution $\overline{\theta}^*$ takes the form,

$$\begin{aligned}\overline{\theta}^* =& \left(k_1^2 - s^2\right)\left(A_1\mathrm{e}^{\mu_1 x} + A_2\mathrm{e}^{-\mu_1 x}\right) + \left(k_2^2 - s^2\right)\left(A_3\mathrm{e}^{\mu_2 x} + A_4\mathrm{e}^{-\mu_2 x}\right)\\ &-\sqrt{\frac{\pi}{2}} \frac{\varepsilon b \mathrm{e}^{-a|q|} \sinh bx}{a\left(b^2-\mu_1^2\right)\left(b^2-\mu_2^2\right)}\end{aligned} \tag{26}$$

By the same reasoning, we can write

$$\begin{aligned}\overline{e}^* =& k_1^2\left(A_1\mathrm{e}^{\mu_1 x} + A_2\mathrm{e}^{-\mu_1 x}\right) + k_2^2\left(A_3\mathrm{e}^{\mu_2 x} + A_4\mathrm{e}^{-\mu_2 x}\right)\\ &-\sqrt{\frac{\pi}{2}}\left\{\frac{\varepsilon\left(b^2-q^2\right) b\mathrm{e}^{-a|q|}\sinh bx}{a\left(b^2-\mu^2\right)\left(b^2-\mu_1^2\right)\left(b^2-\mu_2^2\right)} - \frac{b\mathrm{e}^{-a|q|}\sinh bx}{sa\left(b^2-\mu^2\right)}\right\}\end{aligned} \tag{27}$$

where $\mu = \sqrt{q^2 + s^2}$.

Taking the Fourier transform for both sides of Eq. 15, we get

$$\left(\mathrm{D}^2 - q^2 - \beta^2 s^2\right)\overline{u}^* = \left(1-\beta^2\right)\mathrm{D}\overline{e}^* + \beta^2\mathrm{D}\overline{\theta}^* - \beta^2\overline{F}_x^*. \tag{28}$$

Substituting from Eqs. 26 and 27 into the right-hand side of Eq. 28, we obtain the following equation satisfied by $\overline{u}^*$:

$$\begin{aligned}\left(\mathrm{D}^2 - q^2 - \beta^2 s^2\right)\overline{u}^* =& \sqrt{\frac{\pi}{2}}\frac{\left(\beta^2\mu^2 - b^2\right)\mathrm{e}^{-a|q|}\cosh bx}{sa\left(b^2-\mu^2\right)} + \mu_1\left(k_1^2 - \beta^2 s^2\right)\\ &\times\left(A_1\mathrm{e}^{\mu_1 x} - A_2\mathrm{e}^{-\mu_1 x}\right) + \mu_2\left(k_2^2 - \beta^2 s^2\right)\\ &\times\left(A_3\mathrm{e}^{\mu_2 x} - A_4\mathrm{e}^{-\mu_2 x}\right)\\ &-\sqrt{\frac{\pi}{2}}\frac{\varepsilon b^2\left(b^2-\mu_3^2\right)\mathrm{e}^{-a|q|}\cosh bx}{a\left(b^2-\mu^2\right)\left(b^2-\mu_1^2\right)\left(b^2-\mu_2^2\right)}\end{aligned} \tag{29}$$

The solution $\overline{u}^*$ of Eq. 29 has the form,

$$\begin{aligned}\overline{u}^* =& \frac{\left(\beta^2\mu^2 - b^2\right)\mathrm{e}^{-a|q|}\cosh bx}{sa\left(b^2-\mu^2\right)\left(b^2-\mu_3^2\right)}\sqrt{\frac{\pi}{2}} + \mu_1\left(A_1\mathrm{e}^{\mu_1 x} - A_2\mathrm{e}^{-\mu_1 x}\right)\\ &+\mu_2\left(A_3\mathrm{e}^{\mu_2 x} - A_4\mathrm{e}^{-\mu_1 x}\right) + B_1 ez^{\mu_3 x} + B_2\mathrm{e}^{-\mu_3 x}\\ &-\sqrt{\frac{\pi}{2}}\frac{\varepsilon b^2\mathrm{e}^{-a|q|}\cosh bx}{a\left(b^2-\mu^2\right)\left(b^2-\mu_1^2\right)\left(b^2-\mu_2^2\right)}\end{aligned} \tag{30}$$

where $\mu_3 = \sqrt{q^2 + \beta^2 s^2}$ and $B_i = B_i\,(q, s)$, and $i = 1, 2$ are parameters depending on q and s.

Applying the exponential Fourier transform with respect to y to both sides of Eq. 20, we get

$$\overline{v}* = \frac{1}{\mathrm{i}q}\left(\overline{e} * - \mathrm{D}\overline{u}*\right). \tag{31}$$

Substituting from Eqs. 27 and 30 into the right-hand side of Eq. 31, we obtain

$$\begin{aligned}\overline{v}^* = \mathrm{i}q \Bigg\{ & \frac{\mu_3}{q^2}\left(B_1 \mathrm{e}^{\mu_3 x} - B_2 \mathrm{e}^{-\mu_3 x}\right) + \sqrt{\frac{\pi}{2}}\,\frac{\mathrm{e}^{-a|q|}\left(\beta^2 - 1\right) b \sinh bx}{sa\left(b^2 - \mu^2\right)\left(b^2 - \mu_3^2\right)} \\ & + A_1 \mathrm{e}^{\mu_1 x} + A_2 \mathrm{e}^{-\mu_1 x} + A_3 \mathrm{e}^{\mu_2 x} + A_4 \mathrm{e}^{-\mu_2 x} \\ & - \sqrt{\frac{\pi}{2}}\,\frac{\varepsilon \mathrm{e}^{-a|q|} b \sinh bx}{a\left(b^2 - \mu^2\right)\left(b^2 - \mu_1^2\right)\left(b^2 - \mu_2^2\right)} \Bigg\}\end{aligned} \tag{32}$$

Taking the exponential Fourier transform to both sides of Eqs. 19a, 19c, and 21, we obtain upon using Eqs. 26, 27, 30, and 32, the transforms of the components of the stress tensor in the form:

$$\begin{aligned}\overline{\sigma}^*_{xx} = {} & 2\mu_3\left(B_1 \mathrm{e}^{\mu_3 x} - B_2 \mathrm{e}^{-\mu_3 x}\right) \\ & + \sqrt{\frac{\pi}{2}}\,\frac{b\mathrm{e}^{-a|q|}\left[2q^2\left(\beta^2 - 1\right) + \beta^2\left(\mu_3^2 - b^2\right)\right] \sinh bx}{sa\left(b^2 - \mu^2\right)\left(b^2 - \mu_3^2\right)} \\ & + \left(\beta^2 s^2 + 2q^2\right)\Bigg\{ A_1 \mathrm{e}^{\mu_1 x} + A_2 \mathrm{e}^{-\mu_1 x} + A_3 \mathrm{e}^{\mu_2 x} + A_4 \mathrm{e}^{-\mu_2 x} \\ & - \sqrt{\frac{\pi}{2}}\,\frac{\varepsilon \mathrm{e}^{-a|q|} b \sinh bx}{a\left(b^2 - \mu^2\right)\left(b^2 - \mu_1^2\right)\left(b^2 - \mu_2^2\right)} \Bigg\},\end{aligned} \tag{33}$$

$$\begin{aligned}\overline{\sigma}^*_{xy} = \mathrm{i}q \Bigg\{ & \frac{2q^2 + \beta^2 s^2}{q^2}\left(B_1 \mathrm{e}^{\mu_3 x} + B_2 \mathrm{e}^{-\mu_3 x}\right) \\ & + \sqrt{\frac{\pi}{2}}\,\frac{\mathrm{e}^{-a|q|}\left[\beta^2\left(b^2 + \mu^2\right) - 2b^2\right] \cosh bx}{sa\left(b^2 - \mu^2\right)\left(b^2 - \mu_3^2\right)} \\ & + \Bigg(2\mu_1\left[A_1 \mathrm{e}^{\mu_1 x} - A_2 \mathrm{e}^{-\mu_1 x}\right] + 2\mu_2\left[A_3 \mathrm{e}^{\mu_2 x} - A_4 \mathrm{e}^{-\mu_2 x}\right] \\ & - \sqrt{\frac{\pi}{2}}\,\frac{2\varepsilon \mathrm{e}^{-a|q|} b^2 \cosh bx}{a\left(b^2 - \mu^2\right)\left(b^2 - \mu_1^2\right)\left(b^2 - \mu_2^2\right)} \Bigg)\Bigg\}\end{aligned} \tag{34}$$

and $\overline{q}^*_x$ takes the form,

$$
\begin{aligned}
(1+\tau_0 s)\,\overline{q}_x^* = & -\mu_1\left(k_1^2-s^2\right)\left(A_1\mathrm{e}^{\mu_1 x}-A_2\mathrm{e}^{-\mu_1 x}\right)\\
& -\mu_2\left(k_2^2-s^2\right)\left(A_3\mathrm{e}^{\mu_2 x}-A_4\mathrm{e}^{-\mu_2 x}\right)\\
& -\sqrt{\frac{\pi}{2}}\,\frac{\varepsilon b^2\mathrm{e}^{-a|q|}\cosh bx}{a\left(b^2-\mu_1^2\right)\left(b^2-\mu_2^2\right)}
\end{aligned} \tag{35}
$$

The boundary conditions, Eq. 14, in the transformed domain, respectively, take the form,

$$\overline{\theta}^*(l,q,s)=\sqrt{\frac{2}{\pi}}\,\frac{\theta_0\sin qc}{qs}, \tag{36a}$$

$$\overline{\sigma}_{xx}^*(l,q,s)=0, \tag{36b}$$

$$\overline{\sigma}_{xy}^*(l,q,s)=0, \tag{36c}$$

$$\overline{u}^*(-l,q,s)=0, \tag{36d}$$

$$\overline{v}^*(-l,q,s)=0, \tag{36e}$$

$$\overline{q}_x^*(-l,q,s)=0, \tag{36f}$$

Using the boundary conditions, Eqs. 36a–36f, to evaluate the parameters A_1, A_2, A_3, A_4, B_1, and B_2, we obtain the following system of linear equations:

$$
\begin{aligned}
&\left(k_1^2-s^2\right)\left(A_1\mathrm{e}^{\mu_1 l}+A_2\mathrm{e}^{-\mu_1 l}\right)+\left(k_2^2-s^2\right)\left(A_3\mathrm{e}^{\mu_2 l}+A_4\mathrm{e}^{-\mu_2 l}\right)\\
&=\sqrt{\frac{2}{\pi}}\,\frac{\theta_0\sin qc}{qs}+\sqrt{\frac{\pi}{2}}\,\frac{\varepsilon b\mathrm{e}^{-a|q|}\sinh bl}{a\left(b^2-\mu_1^2\right)\left(b^2-\mu_2^2\right)},
\end{aligned} \tag{37}
$$

$$
\begin{aligned}
&A_1\mathrm{e}^{\mu_1 l}+A_2\mathrm{e}^{-\mu_1 l}+A_3\mathrm{e}^{\mu_2 l}+A_4\mathrm{e}^{-\mu_2 l}+\frac{2\mu_3\left(B_1\mathrm{e}^{\mu_3 l}-B_2\mathrm{e}^{-\mu_3 l}\right)}{\left(\beta^2 s^2+2q^2\right)}\\
&=\sqrt{\frac{\pi}{2}}\,\frac{\varepsilon\mathrm{e}^{-a|q|}b\sinh bl}{a\left(b^2-\mu^2\right)\left(b^2-\mu_1^2\right)\left(b^2-\mu_2^2\right)}\\
&\quad-\sqrt{\frac{\pi}{2}}\,\frac{b\mathrm{e}^{-a|q|}\left[2q^2\left(\beta^2-1\right)+\beta^2\left(\mu_3^2-b^2\right)\right]\sinh bl}{sa\left(\beta^2 s^2+2q^2\right)\left(b^2-\mu^2\right)\left(b^2-\mu_3^2\right)},
\end{aligned} \tag{38}
$$

$$
\begin{aligned}
&\mu_1\left(A_1\mathrm{e}^{\mu_1 l}-A_2\mathrm{e}^{-\mu_1 l}\right)+\mu_2\left(A_3\mathrm{e}^{\mu_2 l}-A_4\mathrm{e}^{-\mu_2 l}\right)\\
&+\frac{2q^2+\beta^2 s^2}{2q^2}\left(B_1\mathrm{e}^{\mu_3 l}+B_2\mathrm{e}^{-\mu_3 l}\right)=\sqrt{\frac{\pi}{2}}\,\frac{\varepsilon b\mathrm{e}^{-a|q|}\cosh bl}{a\left(b^2-\mu^2\right)\left(b^2-\mu_1^2\right)\left(b^2-\mu_2^2\right)}\\
&\qquad-\sqrt{\frac{\pi}{2}}\,\frac{\mathrm{e}^{-a|q|}\left[\beta^2\left(b^2+\mu^2\right)-2b^2\right]\cosh bl}{2sa\left(b^2-\mu^2\right)\left(b^2-\mu_3^2\right)},
\end{aligned} \tag{39}
$$

$$
\begin{aligned}
&\mu_1\left(A_1\mathrm{e}^{\mu_1 l}-A_2\mathrm{e}^{-\mu_1 l}\right)+\mu_2\left(A_3\mathrm{e}^{\mu_2 l}-A_4\mathrm{e}^{-\mu_2 l}\right)+B_1\mathrm{e}^{-\mu_3 l}+B_2\mathrm{e}^{\mu_3 l}\\
&=\sqrt{\frac{\pi}{2}}\,\frac{\varepsilon b^2\mathrm{e}^{-a|q|}\cosh bl}{a\left(b^2-\mu^2\right)\left(b^2-\mu_1^2\right)\left(b^2-\mu_2^2\right)}-\sqrt{\frac{\pi}{2}}\,\frac{\left(\beta^2\mu^2-b^2\right)\mathrm{e}^{-a|q|}\cosh bl}{sa\left(b^2-\mu^2\right)\left(b^2-\mu_3^2\right)},
\end{aligned} \tag{40}
$$

$$A_1 \mathrm{e}^{-\mu_1 l} + A_2 \mathrm{e}^{\mu_1 l} + A_3 \mathrm{e}^{-\mu_2 l} + A_4 \mathrm{e}^{\mu_2 l} + \frac{\mu_3}{q^2}\left(B_1 \mathrm{e}^{-\mu_3 l} - B_2 \mathrm{e}^{\mu_3 l}\right)$$
$$= -\sqrt{\frac{\pi}{2}}\,\frac{\varepsilon b \mathrm{e}^{-a|q|}\sinh bl}{a\left(b^2-\mu^2\right)\left(b^2-\mu_1^2\right)\left(b^2-\mu_2^2\right)} + \sqrt{\frac{\pi}{2}}\,\frac{b\mathrm{e}^{-a|q|}\left(\beta^2-1\right)\sinh bl}{sa\left(b^2-\mu^2\right)\left(b^2-\mu_3^2\right)}, \tag{41}$$

$$\mu_1\left(k_1^2-s^2\right)\left(A_1\mathrm{e}^{-\mu_1 l}+A_2\mathrm{e}^{\mu_1 l}\right)+\mu_2\left(k_2^2-s^2\right)\left(A_3\mathrm{e}^{-\mu_2 l}+A_4\mathrm{e}^{\mu_2 l}\right)$$
$$= \sqrt{\frac{\pi}{2}}\,\frac{\varepsilon b\mathrm{e}^{-a|q|}\cosh bl}{a\left(b^2-\mu_1^2\right)\left(b^2-\mu_2^2\right)}. \tag{42}$$

Solution of the above system of linear equations gives the unknown parameters A_1, A_2, A_3, A_4, B_1, and B_2. This completes the solution of the problem in the transformed domain.

4 Inversion of the Double Transform

We shall now outline the numerical inversion method used to find the solution in the physical domain. Let $\overline{f}^*(x,q,s)$ be the Laplace–Fourier transform of a function $f(x,y,t)$. First, we use the inversion formula of the Fourier transform mentioned earlier to obtain a Laplace expression $\overline{f}(x,y,s)$ of the form,

$$\overline{f}(x,y,s) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \overline{e}^{\mathrm{i}qy}\,\overline{f}^*(x,q,s)\mathrm{d}q$$
$$= \sqrt{\frac{2}{\pi}}\int_{0}^{\infty}\left[\cos(qy)\,\overline{f}_{\mathrm{e}}^*(x,q,s)+\sin(qy)\,\overline{f}_{\mathrm{o}}^*(x,q,s)\right]\mathrm{d}q$$

where $\overline{f}_{\mathrm{e}}^*$ and $\overline{f}_{\mathrm{o}}^*$ denote the even and odd parts of $\overline{f}^*(x,q,s)$, respectively.

The complex inversion formula for Laplace transforms can be written as [16,18]

$$f(x,y,t) = \frac{1}{2\pi\mathrm{i}}\int_{d-\mathrm{i}\infty}^{d+\mathrm{i}\infty} \mathrm{e}^{st}\,\overline{f}(x,y,s)\mathrm{d}s$$

where c is an arbitrary real number greater than all the real parts of the singularities of $\overline{f}(x,q,s)$. Taking $s = d + \mathrm{i}r$, the above integral takes the form:

$$f(x,y,t) = \frac{\mathrm{e}^{dt}}{2\pi}\int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}tr}\,\overline{f}(x,y,d+\mathrm{i}r)\,\mathrm{d}r$$

Expanding the function $h\,(x, q, t) = \exp\,(-\mathrm{d}t)\, f\,(x, y, t)$ in a Fourier series in the interval $[0, 2l]$, we obtain the approximate formula [18],

$$f\,(x, y, t) = f_\infty\,(x, y, t) + E_\mathrm{D}$$

where

$$f_\infty\,(x, y, t) = \frac{1}{2}c_0\,(x, y, t) + \sum_{k=1}^{\infty} c_k\,(x, y, t) \quad \text{for} \quad 0 \le t \le 2l \tag{43}$$

and

$$c_k\,(x, y, t) = \frac{\mathrm{e}^{dt}}{L}\mathrm{Re}\left[\mathrm{e}^{\frac{\mathrm{i}k\pi t}{L}}\overline{f}\left(x, y, d + \frac{\mathrm{i}k\pi}{L}\right)\right], \quad k = 0, 1, 2, \ldots \tag{44}$$

The discretization error E_D can be made arbitrarily small by choosing the constant c sufficiently large [18].

Since the infinite series in Eq. 43 can only be summed up to a finite number N of terms, the approximate value of $f(x, y, t)$ becomes

$$f_N(x, y, t) = \frac{1}{2}c_0(x, y, t) + \sum_{k=1}^{N} c_k(x, y, t) \quad \text{for} \quad 0 \le t \le 2l \tag{45}$$

Using the above formula to evaluate $f(x, y, t)$, we introduce a truncation error E_T that must be added to the discretization error to produce the total approximation error.

Two methods are used to reduce the total error. First, the 'Korrecktur' method [18] is used to reduce the discretization error. Next, the ε-algorithm is used to reduce the truncation error and hence to accelerate convergence.

The Korrecktur method uses the following formula to evaluate the function $f(x, y, t)$:

$$f\,(x, y, t) = f_\infty\,(x, y, t) - \mathrm{e}^{-2cL} f_\infty\,(x, y, 2l + t) + E'_\mathrm{D}$$

where the $\left|E'_\mathrm{D}\right| \ll |E_\mathrm{D}|$ [18].

Thus, the approximate value of $f(x, y, t)$ becomes

$$f_{NK}\,(x, y, t) = f_N\,(x, y, t) - \mathrm{e}^{-2dL} f_{N'}\,(x, y, 2l + t) \tag{46}$$

N' is an integer such that $N' < N$.

We shall now describe the ε-algorithm that is used to accelerate the convergence of the series in Eq. 45. Let N be an odd natural number, and let

$$s_m\,(x, y, t) = \sum_{k=1}^{m} c_k\,(x, y, t)$$

be the sequence of partial sums of Eq. 45. We define the ε-sequence by

$$\varepsilon_{0,m} = 0 \quad \text{and} \quad \varepsilon_{1,m} = s_m,$$

and

$$\varepsilon_{p+1,m} = \varepsilon_{p-1,m+1} + \frac{1}{\varepsilon_{p,m+1} - \varepsilon_{p,m}}, \quad p = 1, 2, 3, \ldots$$

It can be shown that [18] the sequence,

$$\varepsilon_{1,1}, \varepsilon_{3,1}, \varepsilon_{5,1}, \ldots, \varepsilon_{N,1}$$

converges to $f^* (x, q, t) + E_D - c_0/2$ faster than the sequence of partial sums $s_m, m = 1, 2, 3, \ldots$

The actual procedure used to invert the Laplace transforms consists of using Eq. 46 together with the ε-algorithm. The values of d and l are chosen according to the criteria outlined in [18].

5 Numerical Results

The above evaluations were applied to copper material, for the purpose of numerical evaluation, whose constants are shown in Table 1.

The computations were performed for three values of dimensionless time, namely, $t = 0.05, 0.1$, and 0.2. The numerical technique outlined above was used to obtain the temperature, displacement, and the normal stress distributions that are shown in Figs. 1, 2, and 3, respectively. The displacement distribution is continuous for all values of x. The temperature has one discontinuity at each value of the dimensionless time, namely, at $x = 0.1237$ for $t = 0.05$, at $x = -0.1793$ for $t = 0.10$, and at $x = -0.4819$ for $t = 0.20$. The stress distributions have two discontinuities for each value of the dimensionless time, namely, at $x = 0.1237$ and at $x = 0.4415$, at $x = -0.1793$ and $x = 0.3826$, and at $x = -0.4819$ and $x = 0.3422$ for $t = 0.05, t = 0.10$, and $t = 0.20$, respectively.

Table 1 Constants of the problem for copper

$\alpha_t = 1.78 \times 10^{-5}\ \text{K}^{-1}$	$k = 386\ \text{W}\cdot\text{m}^{-1}\cdot\text{K}^{-1}$	$c_E = 383.1\ \text{J}\cdot\text{kg}^{-1}\cdot\text{K}^{-1}$
$\lambda = 7.76 \times 10^{10}\ \text{N}\cdot\text{m}^2$	$\mu = 3.86 \times 10^{10}\ \text{N}\cdot\text{m}^2$	$c_1 = 4.158 \times 10^3\ \text{m}\cdot\text{s}^{-1}$
$\rho = 8954\ \text{kg}\cdot\text{m}^{-3}$	$\eta = 8886.73\ \text{s}\cdot\text{m}^{-2}$	$T_0 = 293\ \text{K}$
$\tau_0 = 0.02\ \text{s}$	$\beta^2 = 4$	$\varepsilon = 0.0168$
$a = 1$	$b = 1$	$c = 1$
$\theta_0 = 1\ \text{K}$	$l = 0.5$	

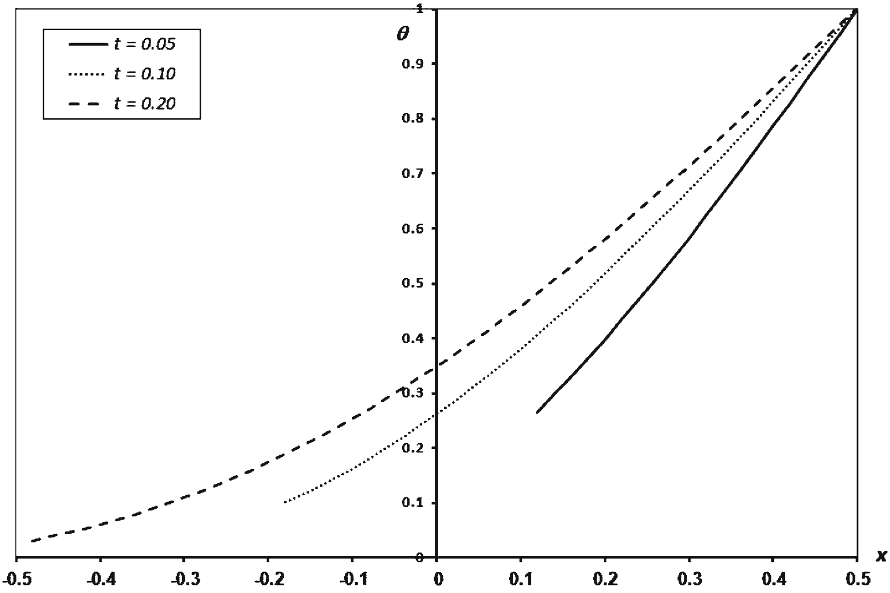

Fig. 1 Distribution of temperature

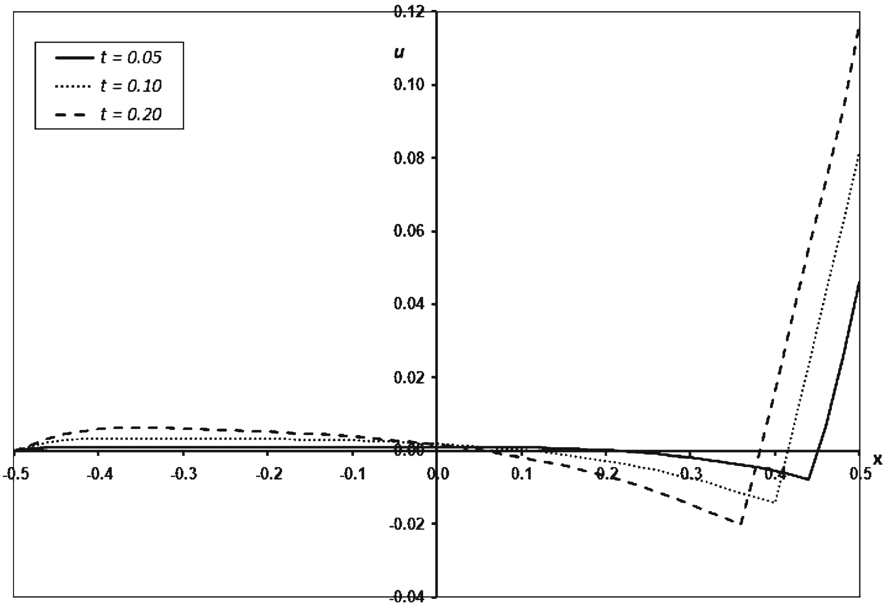

Fig. 2 Distribution of horizontal displacement

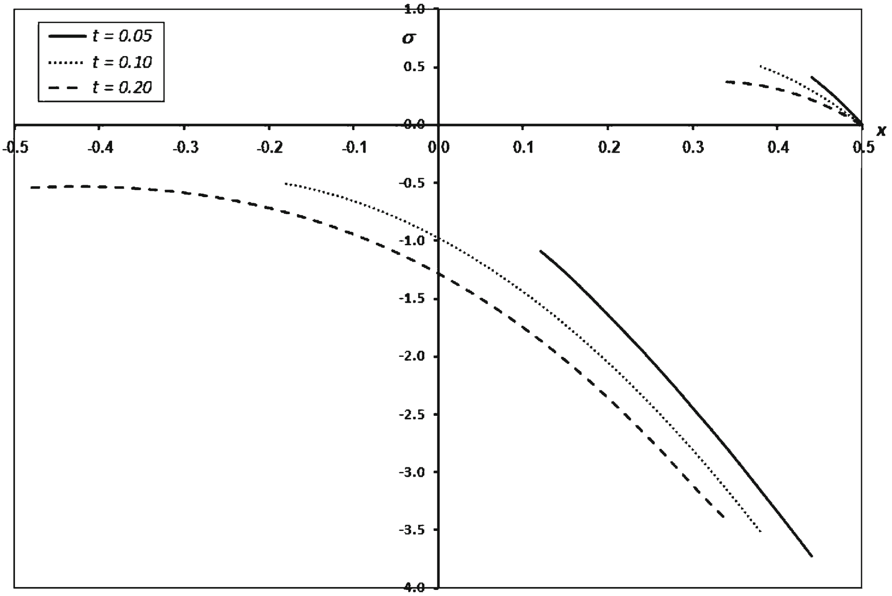

Fig. 3 Distribution of normal stress

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